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Symplectic embedding of a fluid dynamical model

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Abstract

A complete investigation of hidden symmetries present in the d-dimensional fluid dynamical model will be carried out. This will be done in the context of Wess–Zumino (WZ) extension of phase space by using the symplectic embedding formalism. As a consequence, a set of dynamically equivalent symmetries existent in fluid field theory will be discovered. Further, an interesting relation between the WZ symmetries with hidden symmetries (Bazeia D and Jackiw R 1998 *Ann. Phys., NY* **270** 246 (*Preprint* hep-th/9803165); Jackiw R 2000 *Preprint* physics/0010042) will be performed. Indeed, the global *status* of the symmetries will be lifted to a local one.

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1. Introduction

After Bordemann and Hoppe's work [2], the study of scalar fluid field theory has attracted much attention [1, 3–7] over the last few years. In [2], the authors demonstrated that relativistic theories of membranes are integrable systems by reducing the problem to a two-dimensional (2D) fluid dynamics, where the potential term is proportional to the inverse of mass density ($V \propto 1/\rho$). This subject is of wider interest since it also offers connections with the parton model [3], hydrodynamical description of quantum mechanics [8, 9], black-hole cosmology [10] and hydrodynamics of superfluid systems [11]. Most of these investigations are dedicated to finding the solutions of this Galileo invariant system in d-dimensions in connection with the solutions of the relativistic d-brane system in $(d + 1)$-dimensions [1, 4], which is of direct interest to theoretical particle physics.

Some years ago, Jackiw and Bazeia [1] demonstrated that the d-dimensional fluid theory with a specific interaction potential ($V = g/\rho$) presents hidden symmetries: time rescaling and Galileo antiboost invariance. More recently, the presence of the WZ symmetries in fluid theory was investigated as well [12]. Further, in another work, one of us examined the gauge symmetry in the original phase space scenario [13], where WZ fields were not used. However, in both works [12, 13], gauge symmetries were investigated considering a generic interaction potential. Consequently, the global symmetries could not be lifted to local in [12, 13]. In order to fill the lack in this field, we propose in this paper to carry out a complete investigation of the hidden gauge symmetries existent in the fluid dynamics model, with a specific potential term, by using the symplectic embedding formalism [14].

In order for this work to be self-sufficient, it is organized as follows: In section 2, we present a brief review of the symplectic embedding formalism. In section 3, we present a brief review of the fluid dynamics [15]. In section 4, the scalar fluid theory will be analysed from the symplectic point of view [16]. Here, the Dirac brackets among the fields will be computed. In section 5, the symplectic embedding formalism will be used and, as a consequence, gauge-invariant versions of the fluid dynamics model will be obtained. It is important to note that more than one WZ symmetry will be unveiled, showing that this model does not have a unique WZ gauge-invariant description [12], but a family of dynamically equivalent WZ gauge-invariant representations. This will allow an interesting discussion concerning both obvious symmetry (phase symmetry) and hidden symmetry (Galileo antiboost invariance) of the model. Further, the additional symmetries found in [1] (time rescaling and Galileo antiboost invariance) will be investigated from the symplectic embedding point of view. Indeed, the global *status* of these symmetries will be lifted to local. In the last section, we present our concluding observations and final comments.

2. General formalism

In this section, we briefly review the symplectic embedding technique that restores the gauge symmetry. This technique follows Faddeev–Shatashivilli's suggestion [17] and is set up on a contemporary framework to handle constrained models, the symplectic formalism [16].

In order to systemize the symplectic embedding formalism, we consider a general noninvariant mechanical model whose dynamics is governed by a Lagrangian $\mathcal{L}(a_i, \dot{a}_i, t)$, (with $i = 1, 2, \ldots, N$), where a_i and $\dot{a}_i$ are the space and velocity variables, respectively. Note that this model does not result in the loss of generality or physical content. Following the symplectic method the zeroth-iterative first-order Lagrangian one-form is written as

$$\mathcal{L}^{(0)}\,\mathrm{d}t = A_\theta^{(0)}\,\mathrm{d}\xi^{(0)\theta} - V^{(0)}(\xi)\,\mathrm{d}t. \tag{1}$$

The symplectic variables are

$$\xi^{(0)\theta} = \begin{cases} a_i & \text{with } \theta = 1, 2, \ldots, N \\ p_i & \text{with } \theta = N+1, N+2, \ldots, 2N \end{cases} \tag{2}$$

and $A_\theta^{(0)}$ are the canonical momenta and $V^{(0)}$ is the symplectic potential. From the Euler–Lagrange equations of motion, the symplectic tensor is obtained as

$$f_{\theta\beta}^{(0)} = \frac{\partial A_\beta^{(0)}}{\partial \xi^{(0)\theta}} - \frac{\partial A_\theta^{(0)}}{\partial \xi^{(0)\beta}}. \tag{3}$$

When the two-form $f \equiv \frac{1}{2} f_{\theta\beta}\,\mathrm{d}\xi^\theta \wedge \mathrm{d}\xi^\beta$ is singular, the symplectic matrix (3) has a zero mode $(\nu^{(0)})$ that generates a new constraint when contracted with the gradient of the symplectic potential

$$\Omega^{(0)} = \nu^{(0)\theta} \frac{\partial V^{(0)}}{\partial \xi^{(0)\theta}}. \tag{4}$$

This constraint is introduced into the zeroth-iterative Lagrangian one-form equation (1) through a Lagrange multiplier η, generating the next one

$$\begin{aligned}\mathcal{L}^{(1)}\,\mathrm{d}t &= A^{(0)}_\theta\,\mathrm{d}\xi^{(0)\theta} + \mathrm{d}\eta\Omega^{(0)} - V^{(0)}(\xi)\,\mathrm{d}t \\ &= A^{(1)}_\gamma\,\mathrm{d}\xi^{(1)\gamma} - V^{(1)}(\xi)\,\mathrm{d}t \end{aligned} \tag{5}$$

with $\gamma = 1, 2, \ldots, (2N+1)$ and

$$V^{(1)} = V^{(0)}|_{\Omega^{(0)}=0} \qquad \xi^{(1)\gamma} = (\xi^{(0)\theta}, \eta) \qquad A^{(1)}_\gamma = \left(A^{(0)}_\theta, \Omega^{(0)}\right). \tag{6}$$

As a consequence, the first-iterative symplectic tensor is computed as

$$f^{(1)}_{\gamma\beta} = \frac{\partial A^{(1)}_\beta}{\partial \xi^{(1)\gamma}} - \frac{\partial A^{(1)}_\gamma}{\partial \xi^{(1)\beta}}. \tag{7}$$

If this tensor is nonsingular, the iterative process stops and Dirac's brackets among the phase space variables are obtained from the inverse matrix $\left(f^{(1)}_{\gamma\beta}\right)^{-1}$ and, consequently, the Hamilton equation of motion can be computed and solved as well, as discussed in [18]. It is well known that a physical system can be described in terms of a symplectic manifold M, classically at least. From a physical point of view, M is the phase space of the system while a nondegenerate closed two-form f can be identified as being the Poisson bracket. The dynamics of the system is determined just specifying a real-valued function (Hamiltonian) H on phase space, i.e., this real-valued function solves the Hamilton equation, namely

$$\iota(X)f = \mathrm{d}H \tag{8}$$

and the classical dynamical trajectories of the system in phase space are obtained. It is important to mention that if f is nondegenerate, equation (8) has a unique solution. The nondegeneracy of f means that the linear map $\flat : TM \to T^*M$ defined by $\flat(X) := \flat(X)f$ is an isomorphism; due to this, equation (8) is solved uniquely for any Hamiltonian $(X = \flat^{-1}(\mathrm{d}H))$. In contrast, the tensor has a zero mode and a new constraint arises, indicating that the iterative process goes on until the symplectic matrix becomes nonsingular or singular. If this matrix is nonsingular, Dirac's brackets will be determined. In [18], the authors consider in detail the case when f is degenerate, which usually arises when constraints are presented on the system. In which case, (M, f) is called the presymplectic manifold. As a consequence, the Hamilton equation, equation (8), may or may not possess solutions, or possess nonunique solutions. In contrast, if this matrix is singular and the respective zero mode does not generate a new constraint, the system has a symmetry.

After this brief introduction, the symplectic embedding formalism will be systematized. The main idea of this embedding formalism is to introduce extra fields into the model in order to obstruct the solutions of the Hamiltonian equations of motion. It begins with the introduction of two arbitrary functions dependent on the original phase space and WZ variable, namely, $\Psi(a_i, p_i)$ and $G(a_i, p_i, \eta)$, into the first-order Lagrangian one-form as follows:

$$\tilde{\mathcal{L}}^{(0)}\mathrm{d}t = A^{(0)}_\theta\,\mathrm{d}\xi^{(0)\theta} + \Psi\,\mathrm{d}\eta - \tilde{V}^{(0)}(\xi)\,\mathrm{d}t \tag{9}$$

with

$$\tilde{V}^{(0)} = V^{(0)} + G(a_i, p_i, \eta) \tag{10}$$

where the arbitrary function $G(a_i, p_i, \eta)$ is expressed as an expansion in terms of the WZ field, given by

$$G(a_i, p_i, \eta) = \sum_{n=1}^{\infty} \mathcal{G}^{(n)}(a_i, p_i, \eta) \qquad \mathcal{G}^{(n)}(a_i, p_i, \eta) \sim \eta^n \tag{11}$$

and satisfies the following boundary condition

$$G(a_i, p_i, \eta = 0) = 0. \tag{12}$$

The symplectic variables were extended to also contain the WZ variable $\tilde{\xi}^{(0)\tilde{\theta}} = (\xi^{(0)\theta}, \eta)$ (with $\tilde{\theta} = 1, 2, \ldots, 2N + 1$) and the first-iterative symplectic potential becomes

$$\tilde{V}^{(0)}(a_i, p_i, \eta) = V^{(0)}(a_i, p_i) + \sum_{n=1}^{\infty} \mathcal{G}^{(n)}(a_i, p_i, \eta). \tag{13}$$

In this context, the canonical momenta are

$$\tilde{A}^{(0)}_{\tilde{\theta}} = \begin{cases} A^{(0)}_{\theta} & \text{with} \quad \tilde{\theta} = 1, 2, \ldots, 2N \\ \Psi & \text{with} \quad \tilde{\theta} = 2N + 1 \end{cases} \tag{14}$$

and the new symplectic tensor, given by

$$\tilde{f}^{(0)}_{\tilde{\theta}\tilde{\beta}} = \frac{\partial \tilde{A}^{(0)}_{\tilde{\beta}}}{\partial \tilde{\xi}^{(0)\tilde{\theta}}} - \frac{\partial \tilde{A}^{(0)}_{\tilde{\theta}}}{\partial \tilde{\xi}^{(0)\tilde{\beta}}} \tag{15}$$

that is

$$\tilde{f}^{(0)}_{\tilde{\theta}\tilde{\beta}} = \begin{pmatrix} f^{(0)}_{\theta\beta} & f^{(0)}_{\theta\eta} \\ f^{(0)}_{\eta\beta} & 0 \end{pmatrix}. \tag{16}$$

The implementation of the symplectic embedding scheme follows in two steps. The first step is addressed at computing $\Psi(a_i, p_i)$, while the second step is dedicated to the calculation of $G(a_i, p_i, \eta)$. In order to begin with the first step, we impose that this new symplectic tensor $(\tilde{f}^{(0)})$ has a zero-mode $\tilde{\nu}$, consequently, we get the following condition:

$$\tilde{\nu}^{(0)\tilde{\theta}} \tilde{f}^{(0)}_{\tilde{\theta}\tilde{\beta}} = 0. \tag{17}$$

Note that, at this point, f becomes degenerate and, in consequence, we introduce an obstruction to solve, in a unique way, the Hamilton equation of motion given in equation (8). Assuming that the zero-mode $\tilde{\nu}^{(0)\tilde{\theta}}$ is

$$\tilde{\nu}^{(0)} = (\mu^{\theta} \quad 1) \tag{18}$$

and using the relation given in equation (17) together with equation (16), we get a set of equations, namely

$$\mu^{\theta} f^{(0)}_{\theta\beta} + f^{(0)}_{\eta\beta} = 0 \tag{19}$$

where

$$f^{(0)}_{\eta\beta} = \frac{\partial A^{(0)}_{\beta}}{\partial \eta} - \frac{\partial \Psi}{\partial \xi^{(0)\beta}}. \tag{20}$$

Observe that the matrix elements μ^{θ} are chosen in order to disclose a desired gauge symmetry. Note that in this formalism the zero-mode $\tilde{\nu}^{(0)\tilde{\theta}}$ is the gauge symmetry generator. At this point, it deserves mentioning that this characteristic is important because it opens up the possibility of disclosing the desired hidden gauge symmetry from the noninvariant model. It awards some power to the symplectic embedding formalism to deal with noninvariant systems. From relation (17) some differential equations involving $\Psi(a_i, p_i)$ are obtained, equation (19), and after a straightforward computation, $\Psi(a_i, p_i)$ can be determined.

In order to compute $G(a_i, p_i, \eta)$ in the second step, we impose that no more constraints arise from the contraction of the zero-mode $(\tilde{\nu}^{(0)\tilde{\theta}})$ with the gradient of the potential $\tilde{V}^{(0)}(a_i, p_i, \eta)$. This condition generates a general differential equation, which reads as

$$\tilde{\nu}^{(0)\tilde{\theta}} \frac{\partial \tilde{V}^{(0)}(a_i, p_i, \eta)}{\partial \tilde{\xi}^{(0)\tilde{\theta}}} = 0$$

$$\mu^{\theta} \frac{\partial V^{(0)}(a_i, p_i)}{\partial \xi^{(0)\theta}} + \mu^{\theta} \frac{\partial \mathcal{G}^{(1)}(a_i, p_i, \eta)}{\partial \xi^{(0)\theta}} + \mu^{\theta} \frac{\partial \mathcal{G}^{(2)}(a_i, p_i, \eta)}{\partial \xi^{(0)\theta}} + \cdots + \frac{\partial \mathcal{G}^{(1)}(a_i, p_i, \eta)}{\partial \eta} + \frac{\partial \mathcal{G}^{(2)}(a_i, p_i, \eta)}{\partial \eta} + \cdots = 0 \tag{21}$$

that allows us to compute all correction terms $\mathcal{G}^{(n)}(a_i, p_i, \eta)$ in order of η. Note that this polynomial expansion in terms of η is equal to zero, subsequently, whole coefficients for each order in η must be null identically. In view of this, each correction term of order η is determined. For a linear correction term, we have

$$\mu^{\theta} \frac{\partial V^{(0)}(a_i, p_i)}{\partial \xi^{(0)\theta}} + \frac{\partial \mathcal{G}^{(1)}(a_i, p_i, \eta)}{\partial \eta} = 0. \tag{22}$$

For a quadratic correction term, we get

$$\mu^{\theta} \frac{\partial \mathcal{G}^{(1)}(a_i, p_i, \eta)}{\partial \xi^{(0)\theta}} + \frac{\partial \mathcal{G}^{(2)}(a_i, p_i, \eta)}{\partial \eta} = 0. \tag{23}$$

From these equations, a recursive equation for $n \geqslant 2$ is proposed as

$$\mu^{\theta} \frac{\partial \mathcal{G}^{(n-1)}(a_i, p_i, \eta)}{\partial \xi^{(0)\theta}} + \frac{\partial \mathcal{G}^{(n)}(a_i, p_i, \eta)}{\partial \eta} = 0 \tag{24}$$

that allows us to compute the remaining correction terms of order η. This iterative process is successively repeated until equation (21) becomes identically null, consequently, the extra term $G(a_i, p_i, \eta)$ is obtained explicitly. Then, the gauge-invariant Hamiltonian, identified as being the symplectic potential, is obtained as

$$\tilde{\mathcal{H}}(a_i, p_i, \eta) = V^{(0)}(a_i, p_i) + G(a_i, p_i, \eta) \tag{25}$$

and the zero-mode $\tilde{\nu}^{(0)\tilde{\theta}}$ is identified as being the generator of an infinitesimal gauge transformation, given by

$$\delta \tilde{\xi}^{\tilde{\theta}} = \varepsilon \tilde{\nu}^{(0)\tilde{\theta}} \tag{26}$$

where ε is an infinitesimal parameter.

3. The fluid dynamics model

The subject matter considered in this section follows closely that presented in section 2 of [13]. This section begins with the derivation of the Lagrangian that determines the fluid dynamics of interest here. We start with the (linear or nonlinear) Schrödinger theory Lagrangian defined in a d-dimensional ($\mathbf{r}$) space evolving in time (t), which reads as

$$L_{\mathrm{S}} = \int \mathrm{d}^d r \left\{ \mathrm{i}\psi^* \dot{\psi} - \frac{1}{2}(\nabla \psi^*) \cdot (\nabla \psi) - \bar{V}(\psi^* \psi) \right\} \tag{27}$$

with $\bar{V}$ determining any nonlinear interaction. Inserting the representation in terms of mass density ($\rho \equiv \rho(t, \mathbf{r})$) and velocity potential ($\theta \equiv \theta(t, \mathbf{r})$) as in [11], namely,

$$\psi = \rho^{1/2} \, \mathrm{e}^{\mathrm{i}\theta} \tag{28}$$

into the Schrödinger Lagrangian, we get the fluid dynamical model [15] described by the following Lagrangian in d-dimensional ($\mathbf{r}$) space:

$$L = \int \mathrm{d}^d\mathbf{r} \left(\theta\dot{\rho} - \frac{1}{2}\rho\boldsymbol{\nabla}\theta \cdot \boldsymbol{\nabla}\theta - V(\rho) \right) \tag{29}$$

with

$$V(\rho) = \bar{V}(\rho) + \frac{1}{8}\frac{(\boldsymbol{\nabla}\rho)^2}{\rho} \tag{30}$$

which is the hydrodynamical form of the Schrödinger theory [8, 9]. At this point, it is important to note that there is a nontrivial *interaction*, even in the absence of $\bar{V}$. This result can also be obtained from a gauge-fixed formulation of a membrane in Minkowski space [2], through a field-dependent change of variables, for the special case $d = 2$ with the following potential,

$$V(\rho) = \frac{g}{\rho}. \tag{31}$$

The same result was also obtained from a dimensional reduction of a local relativistic field theory [3]. Afterwards, it is obvious that the fluid model described by the Lagrangian, equation (29), with some restrictions on $V(\rho)$, presents Galileo symmetry. As quoted by Bazeia [4], the connection between the fluid model and the membrane and its generalization to the d-brane system only appears under the very specific density-dependent interaction potential ($V = g/\rho$). For completeness, the manifest symmetries and the corresponding generators will be listed in table 1 (see [1] for details).

Table 1. The symmetries and the corresponding generators for the fluid dynamics model.

Energy	$H = \int \mathrm{d}^d r\, \mathcal{E}$
	$\mathcal{E} = \frac{1}{2}\rho\nabla\theta \cdot \nabla\theta + V(\rho)$
Angular momentum	$\mathbf{P} = \int \mathrm{d}^d r\, \mathcal{P}$
	$\mathcal{P} = \rho\nabla\theta = \mathbf{j}$
Galileo boost	$\mathbf{B} = t\mathbf{P} - \int \mathrm{d}^d r\, \mathbf{r}\rho$
	$M = \int \mathrm{d}^d r\, \rho$
Charge	$\rho_\omega = \rho$
	$\theta_\omega = \theta - \omega$
Time dilatation	$D = tH - \int \mathrm{d}^d r\, \rho\theta$
Galileo antiboost	$\mathbf{G} = \int \mathrm{d}^d r \left(\mathbf{r}\mathcal{E} - \frac{1}{2}\rho\nabla\theta^2 \right)$

In [1, 4] it was shown that only under the very specific density-dependent potential, given in equation (31), does the connection between the Galileo invariant system presented in this section, defined either in $d = 2$ or d-space dimensions, and the relativistic membrane and its generalization to the d-brane system in $d = 3$ or $(d + 1)$-space dimensions, appear. In view of this, it is remarkable to note that the additional symmetries present in the Galileo invariant system in $d \geqslant 1$ space dimensions with the interacting potential $V(\rho) = g/\rho$ are also present in the relativistic membrane and its generalization to the d-brane system in $d \geqslant 2$ space dimensions.

4. Symplectic analysis

In this section, the fluid dynamic model will be analysed from the symplectic point of view. Note that this Lagrangian is already in first-order form, so we can write

$$\mathcal{L}^{(0)} = \theta\dot{\rho} - V^{(0)} \tag{32}$$

where the symplectic potential is

$$V^{(0)} = \frac{1}{2}\rho\partial_i\theta\partial^i\theta + V(\rho). \quad (33)$$

The symplectic fields are $\xi^{(0)\beta} = (\rho, \theta)$ with the corresponding canonical momenta given by $A_\rho^{(0)} = \theta$ and $A_\theta^{(0)} = 0$.

The zeroth-iterative symplectic matrix, given by

$$f^{(0)} = \begin{pmatrix} 0 & -\delta^{(d)}(\mathbf{r}-\mathbf{r}') \\ \delta^{(d)}(\mathbf{r}-\mathbf{r}') & 0 \end{pmatrix} \quad (34)$$

is a nonsingular matrix and, consequently, the model is not a gauge-invariant field theory. As settled by the symplectic formalism [16], the Dirac brackets among the phase space fields are acquired from the inverse of the symplectic matrix, namely

$$\{\rho(\mathbf{r}), \theta(\mathbf{r}')\}^* = \delta^{(d)}(\mathbf{r}-\mathbf{r}') \qquad \{\rho(\mathbf{r}), \rho(\mathbf{r}')\}^* = 0 \qquad \{\theta(\mathbf{r}), \theta(\mathbf{r}')\}^* = 0. \quad (35)$$

This completes the noninvariant analysis.

5. The WZ gauge model

At this point, the WZ gauge-invariant version of the fluid theory will be obtained using the symplectic embedding formalism [14]. First, a generic interaction potential will be considered and, later, the specific potential ($V = g/\rho$) will be used. In agreement with the symplectic embedding formalism, two arbitrary functions, Ψ and G, depending on the original phase space fields and the WZ field (η) must be added into the model. The former is introduced into the kinetical sector and the latter into the potential sector of the first-order Lagrangian. The process starts with the computation of Ψ and finishes with the computation of G.

In order to reformulate the model as a gauge-invariant field theory, let us start with the first-order Lagrangian $\mathcal{L}^{(0)}$, equation (32), added with the arbitrary terms (Ψ, G), given by

$$\tilde{\mathcal{L}}^{(0)} = \theta\dot{\rho} + \Psi\dot{\eta} - \tilde{V}^{(0)} \quad (36)$$

with

$$\tilde{V}^{(0)} = \frac{1}{2}\rho\partial_i\theta\partial^i\theta + V(\rho) + G \quad (37)$$

where $\Psi \equiv \Psi(\rho, \theta)$ and $G \equiv G(\rho, \theta, \eta)$ are arbitrary functions to be determined. Now, the symplectic fields are $\tilde{\xi}^{(0)\tilde{\beta}} = (\rho, \theta, \eta)$ while the symplectic matrix is

$$\tilde{f}^{(0)} = \begin{pmatrix} 0 & -\delta^{(d)}(\mathbf{r}-\mathbf{r}') & \frac{\delta\Psi_{\mathbf{r}'}}{\delta\rho(\mathbf{r})} \\ \delta^{(d)}(\mathbf{r}-\mathbf{r}') & 0 & \frac{\delta\Psi_{\mathbf{r}'}}{\delta\theta(\mathbf{r})} \\ -\frac{\delta\Psi_{\mathbf{r}}}{\delta\rho(\mathbf{r}')} & -\frac{\delta\Psi_{\mathbf{r}}}{\delta\theta(\mathbf{r}')} & 0 \end{pmatrix} \quad (38)$$

where $\Psi_{\mathbf{r}} \equiv \Psi(\rho(\mathbf{r}), \theta(\mathbf{r}))$ and $\Psi_{\mathbf{r}'} \equiv \Psi(\rho(\mathbf{r}'), \theta(\mathbf{r}'))$.

As established by the symplectic embedding formalism, the corresponding zero-mode $\tilde{\nu}^{(0)}(\mathbf{r})$ satisfies the relation given in equation (17), that is now rewritten as

$$\int \mathrm{d}^d\mathbf{r}\,\tilde{\nu}^{(0)\tilde{\theta}}(\mathbf{r})\tilde{f}_{\tilde{\theta}\tilde{\beta}}(\mathbf{r}, \mathbf{r}') = 0 \quad (39)$$

which produces a set of equations that allows the determination of Ψ explicitly. At this point, it is very important to note that this embedding formalism reveals the $U(1)$ hidden gauge symmetry on the physical model because the zero mode does not generate a new constraint. Indeed, it determines the arbitrary function Ψ and, consequently, awards the gauge invariant reformulation of the model. After that, the G function will be computed using equation (21).

A remarkable feature of the symplectic embedding technique is that it opens up the possibility of implementing a complete investigation of the WZ gauge symmetries existent on the model just defining the zero mode. In the present case, we could propose eight distinct zero modes in order to explore, apparently, eight different symmetries. With this strategy, we could propose eight dynamically equivalent gauge-invariant versions of the fluid dynamical model given by [12] and we also could lift a global symmetry (phase symmetry) to a local one, indeed, we could obtain a WZ Lagrangian where this symmetry can be obtained in an easy way. It is important to note that the WZ symmetry will be revealed using a generic potential, but at the end of each subsection, we will consider the specific potential ($V = g/\rho$) in order to guarantee the connection between d-dimensional fluid theory and relativistic membrane theory.

5.1. The first hidden symmetry

This subsection begins with the WZ gauge symmetry related to the following zero mode:

$$\tilde{\nu}^{(0)} = (1 \quad 1 \quad -1). \tag{40}$$

Since this zero mode and the symplectic matrix equation (38) must satisfy the gauge symmetry condition, given in equation (39), a set of differential equations is obtained, namely

$$\begin{aligned}
&\int \mathrm{d}^d\mathbf{r}\left(\delta^{(d)}(\mathbf{r}-\mathbf{r}') + \frac{\delta\Psi(\mathbf{r})}{\delta\rho(\mathbf{r}')}\right) = 0\\
&\int \mathrm{d}^d\mathbf{r}\left(-\delta^{(d)}(\mathbf{r}-\mathbf{r}') + \frac{\delta\Psi(\mathbf{r})}{\delta\theta(\mathbf{r}')}\right) = 0\\
&\int \mathrm{d}^d\mathbf{r}\left(\frac{\delta\Psi(\mathbf{r})}{\delta\rho(\mathbf{r}')} + \frac{\delta\Psi(\mathbf{r})}{\delta\theta(\mathbf{r}')}\right) = 0.
\end{aligned} \tag{41}$$

After an integration process, Ψ is determined as

$$\Psi(\mathbf{r}) = \theta(\mathbf{r}) - \rho(\mathbf{r}). \tag{42}$$

In view of this, the symplectic matrix becomes

$$\tilde{f}^{(0)} = \begin{pmatrix} 0 & -1 & -1\\ 1 & 0 & 1\\ 1 & -1 & 0 \end{pmatrix} \delta^{(d)}(\mathbf{r}-\mathbf{r}') \tag{43}$$

which is singular by construction. Due to this, the first-order Lagrangian is

$$\tilde{\mathcal{L}}^{(0)} = \theta\dot{\rho} + (\theta-\rho)\dot{\eta} - \tilde{V}^{(0)} \tag{44}$$

where $\tilde{V}^{(0)}$ is given in equation (37).

Now, let us begin with the second step in order to reformulate the model as a WZ gauge-invariant theory. The zero mode $\tilde{\nu}^{(0)}$ does not produce a constraint when contracted with the gradient of the symplectic potential, namely

$$\int \mathrm{d}^d\mathbf{r}'\,\tilde{\nu}^{(0)\tilde{\beta}}(\mathbf{r})\frac{\delta\tilde{V}^{(0)}(\mathbf{r}')}{\delta\tilde{\xi}^{\tilde{\beta}}(\mathbf{r})} = 0. \tag{45}$$

This expression produces a general differential equation, (21), that allows the computation of whole correction terms of order η enclosed in $G(\rho, \theta, \eta)$, given in equation (11). In order to compute the first correction term of order η, $\mathcal{G}^{(1)}$, we use the relation given in equation (22), written as

$$0 = \int \mathrm{d}^d\mathbf{r}'\left[\frac{1}{2}\partial_i'\theta(\mathbf{r}')\partial'^i\theta(\mathbf{r}')\delta^{(d)}(\mathbf{r}'-\mathbf{r}) + \frac{\delta V(\rho(\mathbf{r}'))}{\delta\rho(\mathbf{r})} + \rho(\mathbf{r}')\partial_i'\theta\partial'^i\delta^{(d)}(\mathbf{r}'-\mathbf{r}) - \frac{\delta\mathcal{G}^{(1)}(\mathbf{r}')}{\delta\eta(\mathbf{r})}\right] \tag{46}$$

where $\partial'_i = \frac{\partial}{\partial \mathbf{r}'}$. After straightforward calculation, the linear correction term of order η is obtained as

$$\mathcal{G}^{(1)} = \frac{1}{2}\eta\partial_i\theta\partial^i\theta + \eta\frac{\delta}{\delta\rho}\int \mathrm{d}^d\mathbf{r}' V(\rho(\mathbf{r}')) + \rho\partial_i\theta\partial^i\eta. \tag{47}$$

Bringing this result back into the symplectic potential, equation (37), we get

$$\tilde{V}^{(0)} = \frac{1}{2}\rho\partial_i\theta\partial^i\theta + V(\rho) + \frac{1}{2}\eta\partial_i\theta\partial^i\theta + \eta\frac{\delta}{\delta\rho}\int \mathrm{d}^d\mathbf{r}' V(\rho(\mathbf{r}')) + \rho\partial_i\theta\partial^i\eta. \tag{48}$$

However, the invariant formulation of the model was not yet obtained because the contraction of the zero mode, equation (40), with the gradient of the symplectic potential above does not generate a null value. Due to this, higher order correction terms of order η must be computed. For the quadratic term, the expression given in equation (23) will be required, which is rewritten as

$$\int \mathrm{d}^d\mathbf{r}' \left[\eta(\mathbf{r}')\frac{\delta^2 V(\rho(\mathbf{r}'))}{\delta\rho^2(\mathbf{r})} + \partial'_i\theta(\mathbf{r}')\partial'^i\eta(\mathbf{r}')\delta^{(d)}(\mathbf{r}'-\mathbf{r}) + \eta(\mathbf{r}')\partial'_i\theta(\mathbf{r}')\partial'^i\delta^{(d)}(\mathbf{r}'-\mathbf{r}) \right.$$
$$\left. + \rho(\mathbf{r}')\partial'^i\eta(\mathbf{r}')\partial'_i\delta^{(d)}(\mathbf{r}'-\mathbf{r}) - \frac{\delta\mathcal{G}^{(2)}(\mathbf{r}')}{\delta\eta(\mathbf{r})}\right] = 0. \tag{49}$$

After a direct calculation, $\mathcal{G}^{(2)}$ is obtained as being

$$\mathcal{G}^{(2)} = \frac{1}{2}\eta^2\frac{\delta^2}{\delta\rho^2}\int \mathrm{d}^d\mathbf{r}' V(\rho(\mathbf{r}')) + \eta\partial_i\theta\partial^i\eta + \frac{1}{2}\rho\partial^i\eta\partial_i\eta. \tag{50}$$

As the second-order correction term is expressed in terms of the potential field (ρ, θ), the contraction of the zero mode with the gradient of the symplectic potential (added with the first- and second-order correction terms) still generates a new constraint, consequently, the next correction term must be computed in order to reveal the symmetry. The third-order correction term ($\mathcal{G}^{(3)}$) is obtained through the following relation:

$$\int \mathrm{d}^d\mathbf{r}' \left[\frac{1}{2}\eta^2(\mathbf{r}')\frac{\delta^3 V(\rho(\mathbf{r}'))}{\delta\rho^3(\mathbf{r})} + \frac{1}{2}\partial'^i\eta(\mathbf{r}')\partial'_i\eta(\mathbf{r}')\delta^{(d)}(\mathbf{r}'-\mathbf{r}) \right.$$
$$\left. + \eta(\mathbf{r}')\partial'_i\eta(\mathbf{r}')\partial'^i\delta^{(d)}(\mathbf{r}'-\mathbf{r}) - \frac{\delta\mathcal{G}^{(3)}(\mathbf{r}')}{\delta\eta(\mathbf{r})}\right] = 0 \tag{51}$$

which leads to the following result for $\mathcal{G}^{(3)}$

$$\mathcal{G}^{(3)} = \frac{1}{6}\eta^3\frac{\delta^3}{\delta\rho^3}\int \mathrm{d}^d\mathbf{r}' V(\rho(\mathbf{r}')) + \frac{1}{2}\eta\partial_i\eta\partial^i\eta. \tag{52}$$

As the contraction of the zero mode, equation (40), with the gradient of the symplectic potential (added with the first-, second- and third-order correction terms) does not generate a null value, the gauge-invariant formulation of the model was not yet achieved. Due to this, higher order correction terms in η must be computed. For the fourth-order correction term, the equation given in equation (24) is used, namely

$$\int \mathrm{d}^d\mathbf{r}' \left[\frac{1}{6}\eta^3(\mathbf{r}')\frac{\delta^4 V(\rho(\mathbf{r}'))}{\delta\rho^4(\mathbf{r})} - \frac{\delta\mathcal{G}^{(4)}(\mathbf{r}')}{\delta\eta(\mathbf{r})}\right] = 0 \tag{53}$$

which after a computation gives

$$\mathcal{G}^{(4)} = \frac{1}{24}\eta^4\frac{\delta^4}{\delta\rho^4}\int \mathrm{d}^d\mathbf{r}' V(\rho(\mathbf{r}')). \tag{54}$$

As $\mathcal{G}^{(4)}$ is written in terms of the $V(\rho)$, the gauge-invariant formulation of the model requires an infinite numbers of WZ terms, which are expressed in a general way for $n > 3$ as

$$\mathcal{G}^{(n)} = \frac{1}{n!}\eta^n \frac{\delta^n}{\delta\rho^n}\int \mathrm{d}^d\mathbf{r}' V(\rho(\mathbf{r}')). \tag{55}$$

Hence, the gauge-invariant first-order Lagrangian is written as

$$\tilde{\mathcal{L}} = \theta\dot{\rho} + (\theta - \rho)\dot{\eta} - \tilde{V}^{(0)} \tag{56}$$

where the symplectic potential is

$$\tilde{V}^{(0)} = \frac{1}{2}(\rho+\eta)(\partial_i\theta)^2 + \frac{1}{2}(\rho+\eta)(\partial_i\eta)^2 + (\rho+\eta)\partial_i\theta\partial^i\eta + V(\rho) + \frac{1}{n!}\eta^n\frac{\delta^n}{\delta\rho^n}\int \mathrm{d}^d\mathbf{r}' V(\rho). \tag{57}$$

Note that the two last terms on the right-hand side of the equation above can be rewritten as

$$\begin{aligned} V(\rho) + \frac{1}{n!}\eta^n\frac{\delta^n}{\delta\rho^n}\int \mathrm{d}^d\mathbf{r}' V(\rho(\mathbf{r}')) &= V(\rho) + \frac{1}{n!}\eta^n\partial_\rho^n V(\rho) \\ &= \left(1 + \frac{1}{n!}\eta^n\partial_\rho^n\right)V(\rho) = \mathrm{e}^{\eta\partial_\rho^n}V(\rho) = V(\rho+\eta). \end{aligned} \tag{58}$$

As a consequence, the symplectic potential becomes

$$\tilde{V}^{(0)} = \tfrac{1}{2}(\rho+\eta)(\partial_i\theta + \partial_i\eta)^2 + V(\rho+\eta). \tag{59}$$

By construction, the contraction of the zero mode ($\tilde{\nu}^{(0)}$) with the gradient of the symplectic potential above does not produce a new constraint, consequently, a WZ symmetry is disclosed.

To complete the gauge-invariant reformulation of the model, the infinitesimal gauge transformation will also be computed. In agreement with the symplectic method, the zero mode $\tilde{\nu}^{(0)}$ is the generator of the infinitesimal gauge transformations ($\delta\mathcal{O} = \varepsilon\tilde{\nu}^{(0)}$). Then,

$$\begin{aligned} \delta\rho(\mathbf{r},t) &= \varepsilon(\mathbf{r},t)\delta^{(d)}(\mathbf{r}-\mathbf{r}') \\ \delta\theta(\mathbf{r},t) &= \varepsilon(\mathbf{r},t)\delta^{(d)}(\mathbf{r}-\mathbf{r}') \\ \delta\eta(\mathbf{r},t) &= -\varepsilon(\mathbf{r},t)\delta^{(d)}(\mathbf{r}-\mathbf{r}') \end{aligned} \tag{60}$$

where $\varepsilon(\mathbf{r},t)$ is an infinitesimal time-dependent parameter. Indeed, under the infinitesimal transformations above, the invariant Hamiltonian ($\tilde{V}^{(0)}$) changes as

$$\delta\tilde{V}^{(0)} = 0. \tag{61}$$

Integrating by parts and using the following field transformations

$$\theta \to \theta - \eta \qquad \rho \to \rho - 2\eta \tag{62}$$

the Lagrangian density, equation (56), becomes

$$\tilde{\mathcal{L}} = -(\rho-\eta)\dot{\theta} - \tfrac{1}{2}(\rho-\eta)(\partial_i\theta)(\partial^i\theta) - V(\rho-\eta) \tag{63}$$

which is the same result obtained in [12]. This can also be rewritten as

$$\tilde{\mathcal{L}} = -\tilde{\rho}\dot{\tilde{\theta}} - \tfrac{1}{2}(\tilde{\rho}\partial_i\tilde{\theta})(\partial^i\tilde{\theta}) - V(\tilde{\rho}) \tag{64}$$

where

$$\tilde{\rho} = \rho - \eta \qquad \tilde{\theta} = \theta \tag{65}$$

which sounds like a Stückelberg field-shifting formalism [19]. Assuming the specific potential $V = g/\rho$, we get

$$\tilde{\mathcal{L}} = -\tilde{\rho}\dot{\tilde{\theta}} - \frac{1}{2}(\tilde{\rho}\partial_i\tilde{\theta})(\partial^i\tilde{\theta}) - \frac{g}{\tilde{\rho}} \tag{66}$$

and, subsequently, the gauge symmetry aspects, which were lost after the implementation of the phase space reduction in the relativistic membrane theory, are now recovered.

5.2. The second hidden symmetry

Now, let us explore a hidden symmetry associated with another zero mode, which is read as

$$\tilde{\nu}^{(0)} = (\,1 \quad 0 \quad -1\,)\,. \tag{67}$$

As the gauge symmetry condition, equation (39), must be satisfied, the following set of differential equations is obtained:

$$\int \mathrm{d}^d\mathbf{r}\frac{\delta\Psi(\mathbf{r})}{\delta\rho(\mathbf{r}')} = 0 \qquad \int \mathrm{d}^d\mathbf{r}\left(-\delta^{(d)}(\mathbf{r}-\mathbf{r}') + \frac{\delta\Psi(\mathbf{r})}{\delta\theta(\mathbf{r}')}\right) = 0 \tag{68}$$

which leads to the following result for Ψ:

$$\Psi(\mathbf{r}) = \theta(\mathbf{r}) \tag{69}$$

with the corresponding symplectic matrix

$$\tilde{f}^{(0)} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \delta^{(d)}(\mathbf{r}-\mathbf{r}'). \tag{70}$$

This matrix is obviously singular and the first-order Lagrangian becomes

$$\tilde{\mathcal{L}}^{(0)} = \theta\dot{\rho} + \theta\dot{\eta} - \tilde{V}^{(0)}. \tag{71}$$

The second step of the symplectic embedding formalism begins with the condition which imposes that the zero mode $\tilde{\nu}^{(0)}$, now given by equation (67), does not produce a constraint when contracted with the gradient of the symplectic potential, namely

$$\int \mathrm{d}^d\mathbf{r}'\tilde{\nu}^{(0)\tilde{\beta}}(\mathbf{r})\frac{\delta\tilde{V}^{(0)}(\mathbf{r}')}{\delta\tilde{\xi}^{\tilde{\beta}}(\mathbf{r})} = 0. \tag{72}$$

This relation generates a general differential equation, given in equation (21), that allows the computation of all correction terms of order η enclosed in $G(\rho, \theta, \eta)$. In order to determine the linear correction term of order η, $\mathcal{G}^{(1)}$, we use the relation given in equation (22), written as

$$\int \mathrm{d}^d\mathbf{r}'\left[\frac{1}{2}\partial_i'\theta(\mathbf{r}')\partial'^i\theta(\mathbf{r}')\delta^{(d)}(\mathbf{r}'-\mathbf{r}) + \frac{\delta V(\rho(\mathbf{r}'))}{\delta\rho(\mathbf{r})} - \frac{\delta\mathcal{G}^{(1)}(\mathbf{r}')}{\delta\eta(\mathbf{r})}\right] = 0. \tag{73}$$

After a calculation, the first correction term of order η is obtained as

$$\mathcal{G}^{(1)} = \frac{1}{2}\eta\partial_i\theta\partial^i\theta + \eta\frac{\delta}{\delta\rho}\int \mathrm{d}^d\mathbf{r}'V(\rho(\mathbf{r}')) \tag{74}$$

and, consequently, the symplectic potential becomes

$$\tilde{V}^{(0)} = \frac{1}{2}\rho\partial_i\theta\partial^i\theta + V(\rho) + \frac{1}{2}\eta\partial_i\theta\partial^i\theta + \eta\frac{\delta}{\delta\rho}\int \mathrm{d}^d\mathbf{r}'V(\rho(\mathbf{r}')). \tag{75}$$

However, the contraction of the zero mode, equation (67), with the gradient of the symplectic potential above generates a non-null value. Due to this, higher order correction terms in η are required. For the quadratic term, the equation given in equation (23) is used, which is written as

$$\int \mathrm{d}^d\mathbf{r}'\left[\eta(\mathbf{r}')\frac{\delta^2 V(\rho(\mathbf{r}'))}{\delta\rho^2(\mathbf{r})} - \frac{\delta\mathcal{G}^{(2)}(\mathbf{r}')}{\delta\eta(\mathbf{r})}\right] = 0. \tag{76}$$

After a calculation, we get

$$\mathcal{G}^{(2)} = \frac{1}{2}\eta^2\frac{\delta^2}{\delta\rho^2}\int \mathrm{d}^d\mathbf{r}'V(\rho(\mathbf{r}')). \tag{77}$$

As $\mathcal{G}^{(2)}$ is expressed in terms of $V(\rho)$, an infinite number of WZ terms appear, which can be denoted in a general way for $n \geqslant 2$ by the following relation:

$$\mathcal{G}^{(n)} = \frac{1}{n!}\eta^n \frac{\delta^n}{\delta\rho^n}\int \mathrm{d}^d\mathbf{r}' V(\rho(\mathbf{r}')). \tag{78}$$

Therefore, the gauge-invariant first-order Lagrangian, after an integration by parts, becomes

$$\tilde{\mathcal{L}} = -(\rho+\eta)\dot{\theta} - \tilde{V}^{(0)} \tag{79}$$

with the symplectic potential

$$\begin{aligned}\tilde{V}^{(0)} &= \frac{1}{2}(\rho+\eta)(\partial_i\theta)^2 + V(\rho) + \frac{1}{n!}\eta^n\frac{\delta^n}{\delta\rho^n}\int \mathrm{d}^d\mathbf{r}' V(\rho(\mathbf{r}')) \\ &= \frac{1}{2}(\rho+\eta)(\partial_i\theta)^2 + V(\rho+\eta)\end{aligned} \tag{80}$$

where we used the relation (58). By construction, the contraction of the zero-mode ($\tilde{\nu}^{(0)}$), equation (67), with the gradient of the symplectic potential above does not produce a new constraint, consequently, a new hidden symmetry is unveiled.

The infinitesimal gauge transformations will be obtained as

$$\delta\rho(\mathbf{r},t) = \varepsilon(\mathbf{r}',t)\delta^{(d)}(\mathbf{r}-\mathbf{r}') \qquad \delta\theta(\mathbf{r},t) = 0 \qquad \delta\eta(\mathbf{r},t) = -\varepsilon(\mathbf{r}',t)\delta^{(d)}(\mathbf{r}-\mathbf{r}') \tag{81}$$

which leave the Hamiltonian invariant ($\delta\tilde{V}^{(0)} = 0$).

Introducing the field transformations, given by

$$\theta \to \theta \qquad \rho \to \rho - 2\eta \tag{82}$$

into the Lagrangian density, equation (79), this becomes the Lagrangian given in equation (63). Using the relation given in equation (65) together with the specific potential $V = g/\rho$, the gauge symmetry aspect of fluid theory is restored as well, which can be verified in equation (66).

5.3. The third hidden symmetry

The symplectic embedding formalism can identify hidden symmetries in a straightforward way and this was presented in a pedestrian way to solve the $\mathcal{G}^{(n)}$ equations in the last two subsections. Due to this, the necessary steps to obtain the arbitrary function $\mathcal{G}$ will be not repeated after this point. Indeed, after this point, we just make known the zero mode and the respective result.

In order to extend the investigation of the hidden symmetries present in the fluid dynamics model, the following zero mode is considered:

$$\tilde{\nu}^{(0)} = (0 \quad 1 \quad -1) \tag{83}$$

which, together with the symplectic matrix, equation (38), and gauge symmetry condition, equation (39), leads to the following set of differential equations for Ψ obtained as

$$\int \mathrm{d}^d\mathbf{r}\left(\delta^{(d)}(\mathbf{r}-\mathbf{r}') + \frac{\delta\Psi(\mathbf{r})}{\delta\rho(\mathbf{r}')}\right) = 0 \qquad \int \mathrm{d}^d\mathbf{r}\frac{\delta\Psi(\mathbf{r})}{\delta\theta(\mathbf{r}')} = 0 \tag{84}$$

which gives the following solution for Ψ:

$$\Psi(\mathbf{r}) = -\rho(\mathbf{r}). \tag{85}$$

From equation (21), we compute the corresponding symplectic potential

$$\begin{aligned}\tilde{V}^{(0)} &= \tfrac{1}{2}\rho\partial_i\theta\partial^i\theta + V(\rho) + \rho\partial_i\theta\partial^i\eta + \tfrac{1}{2}\rho\partial_i\eta\partial^i\eta \\ &= \tfrac{1}{2}\rho(\partial_i\theta + \partial_i\eta)^2 + V(\rho).\end{aligned} \tag{86}$$

Therefore, the gauge-invariant first-order Lagrangian becomes

$$\tilde{\mathcal{L}} = (\theta + \eta)\dot{\rho} - \tilde{V}^{(0)} \tag{87}$$

and the Hamiltonian, equation (86), is invariant under the following infinitesimal gauge transformations:

$$\delta\rho(\mathbf{r}, t) = 0 \qquad \delta\theta(\mathbf{r}, t) = \varepsilon(\mathbf{r}', t)\delta^{(d)}(\mathbf{r} - \mathbf{r}') \qquad \delta\eta(\mathbf{r}, t) = -\varepsilon(\mathbf{r}', t)\delta^{(d)}(\mathbf{r} - \mathbf{r}'). \tag{88}$$

At this stage, it is important to note that with this WZ symmetry it was possible to lift a global symmetry (phase symmetry) to a local one, showing that there is an invariance under local translations of the velocity potential, which also preserves the mass conservation (see table 1).

Using the canonical transformations

$$\theta \to \theta - \eta \qquad \rho \to \rho - \eta \tag{89}$$

the Lagrangian density, equation (87), changes to equation (63). Using equation (65) with the specific interaction potential $V = g/\rho$, the gauge symmetry is restored. One important remark is that the global phase symmetry is lifted to a local one by using WZ fields.

5.4. The fourth hidden symmetry

Here, the following zero mode is considered,

$$\tilde{\nu}^{(0)} = (\,1 \quad -1 \quad -1\,) \tag{90}$$

which due to the symplectic matrix, equation (38), and the gauge symmetry conditions, equations (39) and (21), Ψ and the respective symplectic potential are obtained, namely

$$\begin{aligned}
\Psi(\mathbf{r}) &= \theta(\mathbf{r}) + \rho(\mathbf{r}) \\
\tilde{V}^{(0)} &= \frac{1}{2}(\rho + \eta)(\partial_i\theta)^2 + \frac{1}{2}(\rho + \eta)(\partial_i\eta)^2 - (\rho + \eta)\partial_i\theta\partial^i\eta + V(\rho) + \frac{1}{n!}\eta^n\frac{\delta^n}{\delta\rho^n}\int \mathrm{d}^d\mathbf{r}' V(\rho) \\
&= \frac{1}{2}(\rho + \eta)(\partial_i\theta - \partial_i\eta)^2 + V(\rho + \eta)
\end{aligned} \tag{91}$$

where we used equation (58).

Therefore, the gauge-invariant Lagrangian is obtained as

$$\tilde{\mathcal{L}} = \theta\dot{\rho} + (\theta + \rho)\dot{\eta} - \tilde{V}^{(0)} \tag{92}$$

and the Hamiltonian above is invariant under the following infinitesimal transformations:

$$\begin{aligned}
\delta\rho(\mathbf{r}, t) &= \varepsilon(\mathbf{r}', t)\delta^{(d)}(\mathbf{r} - \mathbf{r}') \\
\delta\theta(\mathbf{r}, t) &= -\varepsilon(\mathbf{r}', t)\delta^{(d)}(\mathbf{r} - \mathbf{r}') \\
\delta\eta(\mathbf{r}, t) &= -\varepsilon(\mathbf{r}', t)\delta^{(d)}(\mathbf{r} - \mathbf{r}').
\end{aligned} \tag{93}$$

From equation (92) and using the following transformations:

$$\theta \to \theta + \eta \qquad \rho \to \rho - 2\eta \tag{94}$$

together with equation (65) and the specific potential, we get the results found in sections 5.1 and 5.3 (see equation (66)).

5.5. The fifth hidden symmetry

At this point, we consider the symmetry related to the following zero-mode:

$$\tilde{\nu}^{(0)} = (-1 \quad 1 \quad -1). \tag{95}$$

This zero mode together with the symplectic matrix, equation (38), must obey the gauge symmetry conditions, equations (39) and (21). Then, we obtain the following results for Ψ and the symplectic potential:

$$\Psi(\mathbf{r}) = -\theta(\mathbf{r}) - \rho(\mathbf{r}) \qquad \tilde{V}^{(0)} = \tfrac{1}{2}(\rho - \eta)(\partial_i\theta + \partial_i\eta)^2 + V(\rho - \eta). \tag{96}$$

In view of this, the gauge-invariant first-order Lagrangian is written as

$$\tilde{\mathcal{L}} = \theta\dot{\rho} - (\theta + \rho)\dot{\eta} - \tilde{V}^{(0)} \tag{97}$$

where the infinitesimal gauge transformations, given by

$$\begin{aligned} \delta\rho(\mathbf{r}, t) &= -\varepsilon(\mathbf{r}', t)\delta^{(d)}(\mathbf{r} - \mathbf{r}') \\ \delta\theta(\mathbf{r}, t) &= \varepsilon(\mathbf{r}', t)\delta^{(d)}(\mathbf{r} - \mathbf{r}') \\ \delta\eta(\mathbf{r}, t) &= -\varepsilon(\mathbf{r}', t)\delta^{(d)}(\mathbf{r} - \mathbf{r}') \end{aligned} \tag{98}$$

leave the Hamiltonian invariant ($\delta\tilde{V}^{(0)} = 0$).

Inserting the field transformations

$$\theta \to \theta - \eta \qquad \rho \to \rho \tag{99}$$

and equation (65) and $V = g/\rho$ into the Lagrangian density, equation (97), we reproduce the results given in sections 5.1 and 5.3.

5.6. The sixth hidden symmetry

Now, a WZ symmetry connected to the following zero mode:

$$\tilde{\nu}^{(0)} = (-1 \quad 1 \quad -1) \tag{100}$$

will be explored. In agreement with the main steps of the symplectic embedding formalism, displayed by equations (38), (39) and (21), Ψ and the gauge-invariant symplectic potential are obtained as

$$\Psi(\mathbf{r}) = -\theta(\mathbf{r}) \qquad \tilde{V}^{(0)} = \tfrac{1}{2}(\rho - \eta)(\partial_i\theta)^2 + V(\rho - \eta). \tag{101}$$

Hence, the gauge-invariant first-order Lagrangian is

$$\tilde{\mathcal{L}} = -(\rho - \eta)\dot{\theta} - \tilde{V}^{(0)}(\rho - \eta) \tag{102}$$

which is the same result obtained in [12], after an integration by parts. Recall that this Lagrangian can be rewritten as the invariant Lagrangian given in equation (66) by using equation (65) and $V = g/\rho$.

The infinitesimal gauge transformations, which leave the Hamiltonian invariant ($\tilde{V}^{(0)}$), are

$$\delta\rho(\mathbf{r}, t) = -\varepsilon(\mathbf{r}', t)\delta^{(d)}(\mathbf{r} - \mathbf{r}') \qquad \delta\theta(\mathbf{r}, t) = 0 \qquad \delta\eta(\mathbf{r}, t) = -\varepsilon(\mathbf{r}', t)\delta^{(d)}(\mathbf{r} - \mathbf{r}'). \tag{103}$$

5.7. The seventh hidden symmetry

An alternative WZ symmetry, associated with the following zero mode

$$\tilde{\nu}^{(0)} = (0 \quad -1 \quad -1) \tag{104}$$

will be unveiled. Again, the main steps of the symplectic potential must be used and, as a consequence, Ψ and the gauge-invariant symplectic potential are determined as

$$\Psi(\mathbf{r}) = \rho(\mathbf{r}) \qquad \tilde{V}^{(0)} = \tfrac{1}{2}\rho(\partial_i\theta - \partial_i\eta)^2 + V(\rho). \tag{105}$$

Hence, the gauge-invariant Lagrangian is

$$\tilde{\mathcal{L}} = (\theta - \eta)\dot{\rho} - \tilde{V}^{(0)} \tag{106}$$

with the corresponding infinitesimal gauge transformations

$$\delta\rho(\mathbf{r},t) = 0 \qquad \delta\theta(\mathbf{r},t) = -\varepsilon(\mathbf{r}',t)\delta^{(d)}(\mathbf{r}-\mathbf{r}') \qquad \delta\eta(\mathbf{r},t) = -\varepsilon(\mathbf{r}',t)\delta^{(d)}(\mathbf{r}-\mathbf{r}') \tag{107}$$

which leave the Hamiltonian invariant. This WZ symmetry, unless with a negative sign, is the same as that considered in section 5.3.

Through the following transformations:

$$\theta \to \theta + \eta \qquad \rho \to \rho - \eta \tag{108}$$

and equation (65) and $V = g/\rho$, the Lagrangian density, equation (106), becomes the Lagrangian given in equation (66). As a consequence, the results of sections 5.1 and 5.3 are obtained.

5.8. The eighth hidden symmetry

In order to accomplish the investigation process of the dynamically hidden symmetries present in the fluid field theory, the following zero mode is considered:

$$\tilde{\nu}^{(0)} = (\,-1 \quad -1 \quad -1\,)\,. \tag{109}$$

Once more, the main steps of the symplectic embedding formalism will be executed. As a consequence, we get

$$\Psi(\mathbf{r}) = -\theta(\mathbf{r}) + \rho(\mathbf{r}) \qquad \tilde{V}^{(0)} = \tfrac{1}{2}(\rho - \eta)(\partial_i\theta - \partial_i\eta)^2 + V(\rho - \eta) \tag{110}$$

where the identity given in equation (58) was used. Hence, the gauge-invariant first-order Lagrangian becomes

$$\tilde{\mathcal{L}} = \theta\dot{\rho} + (-\theta + \rho)\dot{\eta} - \tilde{V}^{(0)} \tag{111}$$

and the infinitesimal gauge transformations, which leave the Hamiltonian invariant, are written as

$$\begin{aligned}\delta\rho(\mathbf{r},t) &= -\varepsilon(\mathbf{r}',t)\delta^{(d)}(\mathbf{r}-\mathbf{r}')\\ \delta\theta(\mathbf{r},t) &= -\varepsilon(\mathbf{r}',t)\delta^{(d)}(\mathbf{r}-\mathbf{r}')\\ \delta\eta(\mathbf{r},t) &= -\varepsilon(\mathbf{r}',t)\delta^{(d)}(\mathbf{r}-\mathbf{r}')\end{aligned} \tag{112}$$

Using the field transformations

$$\theta = \theta + \eta \qquad \rho = \rho \tag{113}$$

together with equation (65) and $V = g/\rho$ in equation (111), the results given in sections 5.1 and 5.3 are obtained.

5.9. Extra symmetries

Now, we propose to shed some light on the question of the origin of the additional symmetries found in [1]. We argue that the time rescaling invariance presented in [1] and in section 3, arises due to the gauge-fixing process of the relativistic theory of membrane [2]. In order to clarify this point, we will use the symplectic embedding formalism. Let us consider the following zero mode

$$\tilde{\nu}^{(0)} = (-\rho \quad \theta \quad -1) \tag{114}$$

which reproduces the same infinitesimal field transformations given in [1]. Following the prescriptions of the symplectic embedding formalism, we obtain

$$\Psi = -\theta\rho \qquad \tilde{V}^{(0)}(\rho, \eta) = \frac{1}{2}\rho(\partial_i\theta)^2\,\mathrm{e}^{\eta} - \frac{g}{\rho}\,\mathrm{e}^{\eta} \tag{115}$$

where the specific interaction potential was considered into the process. Note that a new term, the exponential one, sounds like a Liouville term into the WZ gauge-invariant version of the fluid model. The WZ-invariant Lagrangian density is

$$\tilde{\mathcal{L}} = \theta\dot{\rho} - \theta\rho\dot{\eta} - \frac{1}{2}\rho(\partial_i\theta)^2\,\mathrm{e}^{\eta} + \frac{g}{\rho}\,\mathrm{e}^{\eta} \tag{116}$$

which is invariant under the following infinitesimal gauge transformation:

$$\delta\rho = -\varepsilon\rho \qquad \delta\theta = \varepsilon\theta \qquad \delta\eta = -\varepsilon \tag{117}$$

where ε has no space dependence. This embraces the infinitesimal transformation given in [1].

Now, we consider the following zero mode:

$$\tilde{\nu}^{(0)} = \rho\nabla\theta(-1 \quad 0 \quad 1) \tag{118}$$

which leads to an amazing interpretation for the Galileo antiboost invariance. This zero mode reproduces the infinitesimal field transformations given in [1], given by

$$\delta\rho(t, \mathbf{r}) = -\rho\omega_i\partial^i\theta \qquad \delta\theta(t, \mathbf{r}) = 0 \qquad \delta\eta(t, \mathbf{r}) = \rho\omega_i\partial^i\theta \tag{119}$$

where ω_i is now an infinitesimal parameter. At this point, it is very important to mention that this zero mode, unless with a negative sign and coefficient, is the same as that used to investigate the WZ symmetry in section 5.2. Therefore, this WZ gauge-invariant formulation of the fluid model, whose dynamic is governed by Lagrangian density, given in equation (79), or Hamiltonian density, given in equation (80), is also invariant under the infinitesimal gauge transformations given in equation (119). In section 5.2, we also discuss that this Lagrangian density can be rewritten in an equivalent WZ-invariant Lagrangian description, equation (66), using some canonical transformation, equation (82), and relations, equation (65), together with $V = g/\rho$. This reveals, in a linear fashion, the Galileo antiboost invariance on the WZ gauge-invariant version of the fluid model.

As shown in this section, the symplectic embedding formalism could pick up the extra symmetries proposed in [1] in a linear fashion just enlarging the phase space with the introduction of the WZ fields.

6. Final discussions

We have studied in this paper a complete investigation of the hidden symmetries lying on the fluid field theory [15], which is a theoretical laboratory to study some classical aspects of membrane theory, as shown by Bordemann and Hoppe in [2]. In this paper, the authors demonstrated that the relativistic theory of membranes are integrable systems by reducing the problem to a two-dimensional fluid dynamics. Later, some authors [1, 4] have dedicated themselves to finding the solutions of this Galileo-invariant system in d-dimensions in connection with the solutions of the relativistic d-brane system in $(d+1)$-dimensions, which showed the presence of a hidden dynamical Poincaré symmetry of this nonrelativistic model realized by field dependent diffeomorphism.

In [12], the authors show that both Galileo and Poincaré groups are preserved by the introduction of the WZ fields and that the Galileo and Poincaré groups in the gauged model can be computed from the generators of the nongauged model by using the relation $\tilde{\mathcal{O}} = \mathrm{e}^{-\eta\partial_\rho}\mathcal{O}$. In our study, we re-obtain the gauged fluid model proposed in [12] and seven more gauged versions of the fluid theory, which are dynamically equivalent. In order to clarify this point, we demonstrated that each hidden symmetry investigated in section 5 can be reduced to the symmetry obtained in [12] and section 5.6 through canonical transformations. Further, it was also shown that these sets of equivalent WZ descriptions can be obtained using the Stückelberg field-shifting formalism [19]. In this way, we demonstrated that the d-dimensional fluid field theory has a dynamically equivalent family of WZ gauge-invariant descriptions. As the gauge-invariant versions of the fluid model obtained in this paper are dynamically equivalent and since they can be reduced to the gauged model obtained in [12], they also preserve both Galileo and Poincaré groups. This happens due to the possibility of changing the mass density field, equation (65). Moreover, note that in sections 5.3 and 5.7, an obvious symmetry (phase symmetry) could be identified in a direct way. Indeed, the phase symmetry (global symmetry) had its *status* lifted to a local one. Besides, it was demonstrated that this symmetry belongs to a WZ family of dynamically equivalent gauge descriptions of the fluid model. Another point that is important to mention here is that the WZ gauge symmetry of the d-dimensional fluid theory, with the specific interaction potential $V = g/\rho$, restores the symmetry lost after the phase space reduction process of the relativistic membrane theory.

Further, this work opens up the possibility of verifying extra symmetries [1] in a linear fashion, which seems easier. It was possible to investigate the WZ symmetry connected with the time rescaling conservation, as done in section 5.9. Indeed, both Lagrangian and Hamiltonian, invariant under transformations given in [1], were proposed. Besides, with the WZ embedding of the fluid model, it was possible to demonstrate that the field transformations, equation (119), which are associated with a particular family of WZ symmetry, are also the infinitesimal field transformations produced by the Galileo antiboost generator. Finally, we conclude that after the restoration of the gauge symmetry, the global features of the extra symmetries were naturally obtained in a local one.

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